Announcements

1) In book, subspace test (4.1) has $x+y$ in W instead of $x-y$; either version is fine

Example 1: (not a subspace)

Pick $V = \mathbb{R}^2$

W = graph of $y = 3x + 1$.

W is not a subspace of $\mathbb{R}^2$ since $(0,0)$ is not in W.

Remark: (span and linear independence)

Remember that in $\mathbb{R}^n$, we defined v to be in the span of $\{v_1, v_2, \ldots, v_k\}$ if v is a linear combination of $v_1, v_2, \ldots, v_k$. We said v was linearly independent if it was not in the span.

These definitions carry over unchanged to general vector spaces.

If W is a subspace of V and $S \subseteq W$ is a set of vectors so that every vector v in W is in the span of S, we say S is spanning for W.

Definition: (basis)

A basis for a vector space V is a set $S \subseteq V$ that is both spanning for V and linearly independent.

Definition: (dimension)

Let V be a vector space. The **dimension** of V is the number of vectors in a basis.

If no finite subset of V is a basis, we say V is **infinite dimensional**.

Observation: (subspaces)

If $W \subseteq V$ is a subspace, then if we write "dim" for the dimension,

$$\boxed{\dim(V) \geq \dim(W)}$$

Example 2: $(\mathbb{R}^n$, standard basis$)$

In $\mathbb{R}^n$, let e_i, $1 \le i \le n$, denote the vector with a one in the i^{th} coordinate and zeros in all other coordinates. So in $\mathbb{R}^3$,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Claim: $\{e_i\}_{i=1}^{n}$ is a basis for $\mathbb{R}^n$.

Spanning: If v in $\mathbb{R}^n$ is any vector,

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$ for some

real numbers $v_1, v_2, \ldots, v_n$.

Then

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$$

and hence v is in the span of $\{e_i\}_{i=1}^{n}$.

Linear Independence:

Suppose there are real numbers $a_1, a_2, \ldots, a_n$ with

$$\sum_{i=1}^{n} a_i e_i = \vec{0}.$$

Pick j, $1 \leq j \leq n$, dot product each side with e_j.

We get

$$e_j \cdot \left(\sum_{i=1}^{n} a_i e_i\right) = e_j \cdot \vec{0}$$

$$a_j e_j \cdot e_j \quad = 0$$

Since $e_i \cdot e_j = 0$ for $i \neq j$.

Now $e_j \cdot e_j = 1$, so we get $a_j = 0$. Since j is arbitrary, this holds for all j, $1 \leq j \leq n$.

But: this is not the only basis for $\mathbb{R}^n$!

Example 3: Let $V = \mathbb{R}^3$.

Let $u_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$,

$$u_3 = \begin{bmatrix} 0 \\ 5 \\ 13 \end{bmatrix}.$$

Show $\{u_1, u_2, u_3\}$ is a basis for $\mathbb{R}^3$.

Spanning: Let $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in $\mathbb{R}^3$. We want numbers a_1, a_2, and a_3 with

$$a_1 u_1 + a_2 u_2 + a_3 u_3 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= a_1 \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 5 \\ 13 \end{bmatrix}$$

Solve for a_1, a_2, and a_3.

We get the system of linear equations

$$a_1 + 2a_2 = v_1$$
$$-2a_1 + a_2 + 5a_3 = v_2$$
$$13a_3 = v_3$$

From the last equation,

$$\boxed{a_3 = \frac{v_3}{13}}.$$ Substitute

into second equation to get

$$a_1 + 2a_2 = v_1$$

$$-2a_1 + a_2 + \frac{5}{13}v_3 = v_2$$

$$-2a_1 + a_2 = v_2 - \frac{5}{13}v_3$$

Multiply the 1^{st} equation by 2, add to the second to get

$$5a_2 = 2v_1 + v_2 - \frac{5}{13}v_3,$$

so $$\boxed{a_2 = \frac{2v_1 + v_2 - \frac{5}{13}v_3}{5}}$$

Then from the first equation,

$$a_1 = v_1 - 2a_2$$

$$= \boxed{v_1 - \frac{2\left(2v_1 + v_2 - \frac{5}{13}v_3\right)}{5}}$$

So $\{v_1, v_2, v_3\}$ is spanning for $\mathbb{R}^3$.

Linear Independence

Suppose

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 5 \\ 13 \end{bmatrix}.$$

This is the matrix equation

$$\underbrace{\begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 5 \\ 0 & 0 & 13 \end{bmatrix}}_{A} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\det(A) = 65 \neq 0,$

So A is invertible.

Multiplying both sides by A^{-1}, we get

$$\underbrace{A^{-1}\left(A\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right)}_{= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} = A^{-1}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $a_1 = a_2 = a_3 = 0$!

Example 3: (polynomials)

The monomials

$$\{x^n\}_{n=0}^{\infty}$$ are

a basis for the vector space P.

Spanning: If $p(x) = \sum_{n=0}^{k} a_n x^n$,

we already see it written in terms of the basis vectors!

Linear Independence

Suppose there are

numbers $a_1, a_2, \ldots$

such that for any

$n_1, n_2, \ldots \geq 0,$

$$\sum_{i=0}^{k} a_i x^{n_i} = 0$$

(the zero function)

Then $q_{ij}=0$ for all i, $i \geq 0$, since

x can be any real

number.

Note: These examples say $\dim(\mathbb{R}^n) = n$ and $\dim(P) = \infty$.